

Rumor Spreading on Percolation Graphs

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Abstract

We study the relation between the performance of the randomized rumor spreading (push model) in a d -regular graph G and the performance of the same algorithm in the percolated graph G_p . We show that if the push model successfully broadcast the rumor within T rounds in the graph G then only $(1 + \epsilon)T$ rounds are needed to spread the rumor in the graph G_p when $T = o(pd)$.

1 Introduction

Randomized rumor spreading or *randomized broadcasting* are simple randomized algorithms to spread information in networks. In this work we consider the classical *push model* for rumor spreading, which is described as follows. Initially, one arbitrary vertex knows an information. In the succeeding time steps each informed vertex chooses a neighbor independently and uniformly at random and forward the information to it. The fundamental question is: how many time steps are needed until every vertex of the network has been informed?

The push model has been extensively studied. Most of the papers analyze the runtime of this algorithm on different graph classes. On the complete graph, Frieze and Grimmett [5] proved that *with high probability*¹ $(1 + o(1))(\log_2 n + \log n)$ rounds are necessary and sufficient to inform all vertices. In [3] Feige *et al.* gave general upper bounds holding for any graph and determine the runtime on random graphs. Also, Chierichetti, Lattanzi and Panconesi [2] proved runtime bounds in terms of the conductance and Sauerwald and Stauffer [7] obtain bounds in terms of the vertex expansion for regular graphs.

The starting point of this paper is a recent article by Fountoulakis, Huber and Panagiotou [4] which analyze the push protocol on the Erdős-Rényi random graph $G_{n,p}$ where

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¹With high probability, also denoted w.h.p., refers to an event that holds with probability $1 - o(1)$ as the size of the graph tends to infinity.

$p \gg \frac{\ln n}{n}$. Among other things, they show that the protocol will inform every vertex in $(1 + o(1))(\log_2 n + \ln n)$ steps w.h.p.. One may restate this result by the runtime of randomized broadcasting on a complete graph is essentially not affected by random edge deletions, at least up to the connectivity threshold $p = \ln n/n$.

In this paper we prove a partial extension of this result to the case of arbitrary percolation graphs. Here one starts with some arbitrary graph G and performs edge percolation on it. Under certain conditions, we show that this will not increase the runtime of the protocol by more than a $1 + o(1)$ factor. This suggests that the push protocol is robust against random edge failures, which is a desirable quality for applications.

We need some definitions in order to state our main Theorem. Given a graph $G = (V, E)$, we let G_p denote the random subgraph of G where each edge is removed independently with probability $1 - p$ (the vertex set stays the same). We let $\mathcal{T}_v(G), \mathcal{T}_v(G_p)$ denote the runtimes of the push protocol starting at v over G and G_p (respectively).

Theorem 1.1 *Let $G_n = (V_n, E_n)$ be a sequence of d_n -regular graph where $|V_n| = n \rightarrow \infty$ and $v_n \in V_n$. Suppose that exists T_n such that $\mathcal{T}_{v_n}(G_n) \leq T_n$ with high probability. Then, given any $\epsilon > 0$, if $0 < p_n < 1$ is such that $T_n = o(p_n d_n)$ we have that $\mathcal{T}_{v_n}(G_{n,p_n}) \leq (1 + \epsilon)T_n$ with high probability (here the probability is over the choice of G_{n,p_n} and over the additional randomness of the push protocol).*

One can check in our proofs that the same result holds if G_n has minimum degree d_n . The case where $G_n = K_n$ is complete shows that the condition $T_n = o(p_n d_n)$ cannot be removed in general. We also remark that proving lower bounds for $\mathcal{T}_{v_n}(G_{n,p_n})$ in terms of $\mathcal{T}_{v_n}(G_n)$ is an interesting open problem, but the upper bound we give seems more interesting for applications

1.1 Proof ideas

The proof strategy of [4] relies on the geometry of the Erdős-Rényi graph above the connectivity threshold. We have no such information available in our general setting, and instead rely on a very different proof strategy:

Proof strategy: Construct a coupling of:

$$[\text{push over } G_n] \leftrightarrow [\text{random choice of } G_{n,p} + \text{push over } G_{n,p_n}.]$$

Then show that the runtimes of push over the two graphs are close under the coupling.

It is not hard to sketch a coupling that solves the analogous problem over oriented graphs. Assume $D = (V, F)$ is a n -vertex *digraph*. Define D_p as the random digraph

obtained from D by deleting each oriented edge with probability $1 - p$. We consider a variant of push over D and D_p , where each informed vertex v pushes the rumour along outgoing edges chosen uniformly but *without replacement*. We will assume that all vertices have the same out-degree d and that this modified push protocol over D typically takes $T \ll pd$ steps.

We now couple

$$[\text{push over } D \text{ up to time } T] \leftrightarrow [\text{random choice of } D_p + \text{push over } D_p \text{ up to time } T.]$$

For this we need two Bernoulli (indicator) random variables $A_{v \rightarrow w}$ and $I_{v \rightarrow w}$ for each oriented edge $v \rightarrow w$. All of those variables will be assumed independent, and we take:

$$\mathbb{P}(A_{v \rightarrow w} = 1) = p \text{ and } \mathbb{P}(I_{v \rightarrow w} = 1) = \frac{CT}{pd} \text{ for some } C > 0.$$

Notice that $T \geq \log_2 n$ as the number of informed vertices can only double at each time step. Chernoff bounds (see [1]) imply that, if C is sufficiently large, then w.h.p., for all $v \in V$, the set

$$\mathcal{N}(v) \equiv \{w \in V : v \rightarrow w \in F, A_{v \rightarrow w} I_{v \rightarrow w} = 1\}$$

will have at least T elements, as each of the d out-neighbors of v belongs to this set with probability CT/d . This means we can run modified push over D by having each v select the edges $v \rightarrow w$ with $w \in \mathcal{N}(v)$ in a random order, up to time T . Notice that this gives the right distribution because, conditionally on $|\mathcal{N}(v)| = k \geq T$, $\mathcal{N}(v)$ is uniform over all k -subsets of out-neighbors of v in D .

We now let D_p be the digraph whose oriented edges are the pairs $v \rightarrow w$ with $A_{v \rightarrow w} = 1$. To run modified push on D_p , have each v select the edges $v \rightarrow w$, $w \in \mathcal{N}(v)$, in the same order as in the protocol over D . The key points are that:

- This gives the right distribution because conditionally on D_p and on $|\mathcal{N}(v)| = k$, $\mathcal{N}(v)$ is uniform over k -subsets of the out-neighbors of v in D_p .
- The set of informed vertices in D and D_p coincide up to time T . In particular, if all vertices are informed in D up to time T w.h.p., the same will hold in D_p .

This shows that the modified push protocol cannot take longer in D_p than its typical runtime in D .

As presented, this proof strategy cannot work for non-oriented graphs. The first problem is that the neighbors to be informed in the push protocol are chosen with replacement. This, however, is not hard to deal with; see Proposition 3.1 below.

A second and more serious problem is that, if one tries to copy the above coupling, the events $w \in \mathcal{N}(v)$ and $v \in \mathcal{N}(w)$ will be positively correlated given $\{v, w\} \in G_{n,p_n}$. The solution to this will be to introduce a few *extra steps* in the push protocol over G_{n,p_n} . This

is a kind of sprinkling idea. The upshot will be that the set of neighbors of v chosen in push-over- G_{n,p_n} will dominate the set chosen in push-over- G , but the difference between the two sets will be so small that this will not matter much. (Incidentally, this is where the ϵ in the Theorem comes from.)

1.2 Organization

This paper is organized as follows. In Section 2 we describe our model more formally, introduce some basic notation and prove preliminary results that will be used in the proof of the main result. In Section 3 we present the proof of the Theorem 1.1. Some of the results stated in Section 2 are proved in the Appendix.

2 Preliminaries

Let $G = (V, E)$ be an unweighted, undirect, simple and connected graph, where V is the set of vertices, E the set of edges and $|V|$ denotes the size of the graph. We consider families of graphs $G_n = (V_n, E_n)$ where $|V_n| = n$. For a vertex $v \in V$, $\deg(v)$ denotes the degree of v and $\Gamma(v)$ denotes the set of neighbors of v .

Given some parameter $p \in [0, 1]$, we consider bond percolation in G by removing each edge of G , independently and with probability $1 - p$. The graph obtained from this process is denoted by G_p . Also, $\deg_p(v)$ denotes the degree of v in G_p and $\Gamma_p(v)$ denotes the set of neighbors of v in G_p .

As mentioned before, we consider the randomized broadcasting algorithm called push algorithm. Formally, the process just explained can be described as follows. Let $I(t)$ be the set of vertices informed at time t . Initially $t = 0$ and $I(0) = \{v\}$, for some choice of $v \in V$. While $I(t) \neq V$, each vertex $u \in I(t)$ chooses a neighbor v_u^t independently and uniformly at random. The new informed set is

$$I(t+1) = I(t) \cup \{v_u^t; u \in I(t)\}.$$

The process stops when $I(t) = V$.

We are interested in how many time steps are needed for the process to stop. For this, define $\mathcal{T}_v(G) = \min\{t \in \mathbb{N} | I(t) = V\}$ as the first time step in which every vertex of G has been informed.

A concept used throughout this work is stochastic domination. If X and Y are random variables taking values in \mathbb{R} , we say that X is *stochastically smaller* than Y and denote this by $X \preceq Y$ if $\mathbb{P}(X \geq t) \leq \mathbb{P}(Y \geq t)$ for any $t \in \mathbb{R}$.

If X has distribution μ and Y has distribution ν , then $X \preceq Y$ if and only if

$$\int f d\mu \leq \int f d\nu$$

for any continuous and increasing function f . By approximation, $X \preceq Y$ implies the relation above for all increasing upper semicontinuous function f .

In this paper we write $X \sim Y$ when the random variables X and Y have the same distribution. We let $Geo(p)$, $Be(p)$, $Bin(n, p)$ denote, respectively, the *geometric distribution* with parameter p , the *Bernoulli distribution* with parameter p and the *binomial distribution* with parameters n and p .

The rest of this section is devoted to the study of the negative hypergeometric distribution. A negative hypergeometric random variable X records the waiting time in trials until the r -th success is obtained in repeated random sampling without replacement from a dichotomous population of size N with d successes. In the following we show that a negative hypergeometric random variable is stochastically smaller than a sum of geometric random variables. After that we can bound the probability that the sum of negative hypergeometric distribution deviates above the mean using concentration inequalities for the sum of geometric random variables.

Formally, X has *negative hypergeometric distribution*, denoted by $X \sim NH(N, d, r)$, if

$$\mathbb{P}(X = k) = \frac{\binom{k-1}{r-1} \binom{N-k}{d-r}}{\binom{N}{d}}.$$

The expected value of X is $r \frac{N+1}{d+1}$.

We start studying the relation between geometric and negative hypergeometric random variables.

Lemma 2.1 *For each $j = 1, \dots, k_1$, let X_j be the waiting time until the j th success in trials without replacement from a population of size $k_1 + k_2$ with k_1 possibilities of success (each X_j has distribution $NH(k_1 + k_2, k_1, j)$). Then:*

$$X_1 \preceq Geo\left(\frac{k_1}{k_1 + k_2}\right) \preceq Geo\left(\frac{k_1}{k_1 + k_2} - \frac{j}{k_1 + k_2}\right); \quad (1)$$

$$X_{j+1} - X_j | X_j \preceq Geo\left(\frac{k_1}{k_1 + k_2} - \frac{j}{k_1 + k_2}\right). \quad (2)$$

Proof: Since the number of failures in the population of size $k_1 + k_2$ is k_2 , we have that

$$\begin{aligned} \mathbb{P}(X_1 \geq m) &= \frac{k_2}{k_1 + k_2} \frac{k_2 - 1}{k_1 + k_2 - 1} \cdots \frac{k_2 - (m-2)}{k_1 + k_2 - (m-2)} \\ &\leq \left(\frac{k_2}{k_1 + k_2}\right)^{m-1} \\ &= \mathbb{P}\left(Geo\left(\frac{k_1}{k_1 + k_2}\right) \geq m\right), \end{aligned}$$

where the last inequality follows observing that $\frac{k_2-j}{k_1+k_2-j} \leq \frac{k_2}{k_1+k_2}$. The second inequality in (1) is immediate.

Now to prove (2) observe that conditioned in $X_j = k$ the remaining population has size $k_1 + k_2 - k$ and $k_2 - (k - j)$ failures. Thus,

$$\begin{aligned} \mathbb{P}(X_{j+1} - X_j \geq m | X_j = k) &= \frac{k_2 - (k - j)}{k_1 + k_2 - k} \frac{k_2 - (k - j) - 1}{k_1 + k_2 - k - 1} \dots \frac{k_2 - (k - j) - (m - 2)}{k_1 + k_2 - k - (m - 2)} \\ &\leq \left(\frac{k_2}{k_1 + k_2} + \frac{j}{k_1 + k_2} \right)^{m-1} \\ &= \mathbb{P} \left(\text{Geo} \left(\frac{k_1}{k_1 + k_2} - \frac{j}{k_1 + k_2} \right) \geq m \right), \end{aligned}$$

where the above inequality holds because

$$\begin{aligned} \frac{k_2 - (k - j)}{k_1 + k_2 - k} &= \frac{\frac{k_2}{k_1 + k_2}(k_1 + k_2) - k \frac{k_2}{k_1 + k_2}}{k_1 + k_2 - k} + \frac{-k \frac{k_1}{k_1 + k_2} + j}{k_1 + k_2 - k} \\ &\leq \frac{k_2}{k_1 + k_2} + \frac{j}{k_1 + k_2} \end{aligned}$$

and

$$\frac{k_2 - (k - j) - l}{k_1 + k_2 - k - l} \leq \frac{k_2 - (k - j)}{k_1 + k_2 - k},$$

for $l = 1, \dots, m - 2$. \square

The next lemma enables us to dominate a negative hypergeometric random variable by a sum of geometric random variables. Its proof can be found in the appendix.

Lemma 2.2 *Let X_1, \dots, X_j and Y_1, \dots, Y_j be random variables taking values in \mathbb{R} such that $X_0 \equiv 0$ and $X_{i+1} - X_i | X_i \preceq Y_{i+1}$, for $i = 0, \dots, j - 1$, then*

$$X_j \preceq \tilde{Y}_1 + \dots + \tilde{Y}_j$$

where $\tilde{Y}_i \sim Y_i$ and \tilde{Y}_i is independent of \tilde{Y}_k for $k \neq i$.

We use Lemma 2.1 combined with Lemma 2.2 to obtain that if $X_j \sim NH(k_1 + k_2, k_1, j)$ with $j \leq k_1$, then $X_j \preceq G_1 + \dots + G_j$, where G_1, \dots, G_j are independent geometric random variables with parameter $\frac{k_1}{k_1 + k_2} - \frac{j}{k_1 + k_2}$.

Now, consider the following standard fact about stochastic domination, which is a corollary of the Lemma 2.2.

Corollary 2.1 *Let $X_1, \dots, X_r, Y_1, \dots, Y_r$ be random variables taking values in \mathbb{R} such that $\{X_1, \dots, X_r\}$ and $\{Y_1, \dots, Y_r\}$ are independent. If $X_i \preceq Y_i$ for $i = 1, \dots, r$, then*

$$X_1 + \dots + X_r \preceq Y_1 + \dots + Y_r.$$

If $X_j \sim \text{NH}(k_1 + k_2, k_1, j)$ and $X_l \sim \text{NH}(k_1 + k_2, k_1, l)$ are independent random variables, we can use Lemma 2.1 to guarantee that $X_j + X_l \preceq G_1 + \dots + G_{j+l}$, where G_1, \dots, G_{j+l} are independent geometric random variables with parameter $\frac{k_1}{k_1 + k_2} - \frac{\max\{j, l\}}{k_1 + k_2}$.

At this point, the problem of bounding the probability of the sum of negative hypergeometric random variables deviations above the mean is transformed into the problem of bounding the sum of geometric random variables. In the next lemma we give a bound for the sum of geometric random variables with parameter $1 - o(1)$.

Lemma 2.3 *Given any $\epsilon > 0$ and $C > 1$ there exists $\delta > 0$ such that if G_1, \dots, G_r are independent random variables with distribution geometric with parameter $p \geq 1 - \delta$, then*

$$\mathbb{P}(G_1 + \dots + G_r > (1 + \epsilon)r) \leq \exp(-(C - 1)r).$$

Proof: Begin by taking $\epsilon > 0$ and $C > 1$. If the parameter is $1 - \delta$,

$$\begin{aligned} \mathbb{P}(G_1 + \dots + G_r \geq (1 + \epsilon)r) &\leq \exp\left(-\frac{C}{\epsilon}r\epsilon\right) \mathbb{E}\left[\exp\left(\frac{C}{\epsilon}\sum_{i=1}^r(G_i - 1)\right)\right] \\ &= \exp(-Cr) \left(\sum_{k=1}^{\infty} (1 - \delta) \delta^{k-1} \exp\left(\frac{C}{\epsilon}(k - 1)\right)\right)^r \\ &\leq \exp(-Cr) \left(\sum_{k=1}^{\infty} \left(\delta \exp\left(\frac{C}{\epsilon}\right)\right)^{k-1}\right)^r. \end{aligned}$$

For δ sufficiently small $\delta \exp(\frac{C}{\epsilon}) \leq \delta^{\frac{1}{2}} \leq \frac{1}{2}$ and so

$$\begin{aligned} \exp(-Cr) \left(\sum_{k=1}^{\infty} \left(\delta \exp\left(\frac{C}{\epsilon}\right)\right)^{k-1}\right)^r &= \exp(-Cr) \left(\frac{1}{1 - \delta^{\frac{1}{2}}}\right)^r \\ &\leq \exp(-Cr) \left(1 + 2\delta^{\frac{1}{2}}\right)^r \\ &\leq \exp\left(-\left(C - 2\delta^{\frac{1}{2}}\right)r\right) \\ &\leq \exp(-(C - 1)r). \end{aligned}$$

This gives the result if the parameter of the geometrics is $1 - \delta$. The case $p \geq 1 - \delta$ follows via stochastic domination. \square

3 Proof of Theorem 1.1

We begin by defining another process, called *push without replacement* and denoted by PWR or $\text{PWR}(G)$: at time $t = 0$ an arbitrary vertex knows an information. In the succeeding time steps each informed vertex chooses a neighbor independently and uniformly at random from its neighbors not yet chosen and forwards the information according to this list.

Formally, let $J(t)$ be the set of vertices informed by PWR at time t . Initially $t = 0$ and $J(0) = \{v\}$, for some choice of $v \in V$. While $J(t) \neq V$, each vertex $u \in J(t)$ chooses a neighbor v_u^t independently and uniformly at random in $\Gamma(u) \setminus \{v_u^s; 0 < s < t\}$. The new informed set is

$$J(t+1) = J(t) \cup \{v_u^t; u \in J(t)\}.$$

The process stops when $J(t) = V$.

Define $\mathcal{J}_v(G) = \min\{t \in \mathbb{N} | J(t) = V\}$ as the first time step in which every vertex of G has been informed by the PWR. The next result relates $\mathcal{T}_v(G)$ and $\mathcal{J}_v(G)$. We assume that $I(0) = J(0) = \{v_n\}$ and throughout this section we denote $G = G_n, p = p_n, T = T_n, d = d_n$ and $v = v_n$, we also omit v from $\mathcal{T}_v(G)$ and $\mathcal{J}_v(G)$.

Proposition 3.1 *For any graph G we have that $\mathcal{J}(G) \preceq \mathcal{T}(G)$. Moreover, let $G_n = (V_n, E_n)$ be a sequence of d_n -regular graphs, with $n \rightarrow \infty$, and $T_n = o(d_n)$ such that $\mathcal{J}(G_n) \leq T_n$ with high probability. Then given any $\epsilon > 0$, $\mathcal{T}(G_n) \leq (1 + \epsilon)T_n$ with high probability.*

Proof: Given any graph G , we start building a coupling between the push and the PWR in G as follows. For each $u \in V$, let $\Theta^u = \{\Theta_i^u\}_{i=1}^\infty$ be a sequence of independent random variables with uniform distribution in $\Gamma(u)$. Run the push in G according to the realization of $\{\Theta^u\}_{u \in V}$ in the following sense: the first neighbor to be chosen by a vertex u is Θ_1^u , the second is Θ_2^u and so on.

Now, for $u \in V$, define $X_1^u = 1$ and, for each $k \in \mathbb{N}$,

$$X_k^u = \inf\{m \in \mathbb{N} | \Theta_m^u \notin \{\Theta_{X_1^u}^u, \dots, \Theta_{X_{k-1}^u}^u\}\}.$$

Also, define $U_k^u = \Theta_{X_k^u}^u$. We will use the following fact (proof omitted):

Claim 3.1 *For an arbitrary $u \in V$, let $\deg(u) = s$. The random variable $(U_j^u)_{j=1}^s$ has uniform distribution in the permutations of $\Gamma(u)$. Moreover, for each $k \leq s$, $X_k^u - X_{k-1}^u$ has geometric distribution with parameter $1 - \frac{k-1}{s}$ and the vector $(U_j^u)_{j=1}^s$ and the random variables $X_1^u, X_2^u - X_1^u, \dots, X_s^u - X_{s-1}^u$ are mutually independent.*

Now, run the PWR according to the realization of $\{(U_1^u, \dots, U_{\deg(u)}^u)\}_{u \in V}$ in the following sense: each vertex u informs its neighbors in the order given by the list $(U_1^u, \dots, U_{\deg(u)}^u)$.

With the processes constructed in this way, we have $I(t) \subset J(t)$ for all $t \in \mathbb{N}$, which implies $\mathcal{J}(G) \preceq \mathcal{T}(G)$ and we have the first part of the Lemma.

It remains to prove the second assertion. Let $v = I(0) = J(0)$ and G a d -regular graph with $T = o(d)$, then

$$\begin{aligned}
\mathbb{P}(\mathcal{T}(G) > (1 + \epsilon)T) &= \mathbb{P}(\mathcal{T}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T) \\
&\quad + \mathbb{P}(\mathcal{T}(G) > (1 + \epsilon)T, \mathcal{J}(G) > T) \\
&= \mathbb{P}(\mathcal{T}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T) + o(1) \\
&= \mathbb{P}(\exists w \in V; \mathcal{T}_{v,w}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T) + o(1) \\
&\leq n \cdot \mathbb{P}(\mathcal{T}_{v,w'}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T) + o(1),
\end{aligned}$$

where $\mathcal{T}_{v,w}(G) = \min\{t \in \mathbb{N} | w \in I(t)\}$ is the first time step in which w is informed and w' is defined by the equality $\max_{w \in V} \mathbb{P}(\mathcal{T}_{v,w}(G) > (1 + \epsilon)T) = \mathbb{P}(\mathcal{T}_{v,w'}(G) > (1 + \epsilon)T)$.

Observe that

$$1\{\mathcal{T}_{v,w'}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T\} = f(\{X_k^u - X_{k-1}^u\}_{k=1, u \in V}^{k=d}, \{(U_{X_k^u})_{k=1}^d\}_{u \in V})$$

where f is a measurable function. Recall that $\{X_k^u - X_{k-1}^u | 1 \leq k \leq d, u \in V\}$ and $\{(U_{X_k^u})_{k=1}^d | u \in V\}$ are mutually independent, then by the substitution principle, we have that

$$\begin{aligned}
\mathbb{P}(\mathcal{T}_{v,w'}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T) &= \mathbb{E} \left[\mathbb{E} \left[f(\{X_k^u - X_{k-1}^u\}_{k,u}, \{(U_{X_k^u})_k\}_u) | \{(U_{X_k^u})_k\}_u \right] \right] \\
&= \mathbb{E} [\phi(\{U_{X_k^u}\}_k)_u],
\end{aligned}$$

where $\phi(\{(x_k^u)_k\}_u) = \mathbb{E} [f(\{X_k^u - X_{k-1}^u\}_{k,u}, \{(x_k^u)_k\}_u)]$.

Fixed the realization $\{(U_{X_k^u})_k\}_u = \{(x_k^u)_k\}_u$, the event $\{\mathcal{J}(G) \leq T\}$ implies the existence of a path $\gamma : v = v_0 \sim \dots \sim v_l = w'$ such that $J_{v_0, v_1} + \dots + J_{v_{l-1}, v_l} \leq T$, where $J_{v_j, v_{j+1}}$ is the first time step in which v_j choose v_{j+1} to transmit the information in the PWR. As $\{\mathcal{T}_{v,w'}(G) > (1 + \epsilon)T\}$, defining $T_{v_j, v_{j+1}}$ as the first time step in which v_j choose v_{j+1} to transmit the information in the push in G , we have that $T_{v_0, v_1} + \dots + T_{v_{l-1}, v_l} \geq (1 + \epsilon)T$. So,

$$\mathbb{E} [f(\{X_k^u - X_{k-1}^u\}_{k,u}, \{(x_k^u)_k\}_u)] \leq \mathbb{P}(T_{v_0, v_1} + \dots + T_{v_{l-1}, v_l} \geq (1 + \epsilon)T).$$

To bound the probability above we find the distribution of $T_{v_j, v_{j+1}}$,

$$\begin{aligned}
T_{v_j, v_{j+1}} &= X_{J_{v_j, v_{j+1}}}^{v_j} \\
&= X_1^{v_j} + (X_2^{v_j} - X_1^{v_j}) + \dots + (X_{t_{v_j, v_{j+1}}}^{v_j} - X_{t_{v_j, v_{j+1}}-1}^{v_j}) \\
&\sim G_1^j + \dots + G_{J_{v_j, v_{j+1}}}^j
\end{aligned}$$

where $G_1^j, \dots, G_{J_{v_j, v_{j+1}}}^j$, by the Claim 3.1, are independent random variables such that G_k^j has geometric distribution with parameter $1 - \frac{k-1}{d}$, for each $k = 1, \dots, J_{v_j, v_{j+1}}$.

As $\text{Geo}(1 - \frac{k-1}{d}) \preceq \text{Geo}(1 - \frac{T}{d})$ and $J_{v_0, v_1} + \dots + J_{v_{l-1}, v_l} \leq T$, using Lemma 2.1 we have

$$T_{v_0, v_1} + \dots + T_{v_{l-1}, v_l} \preceq G'_1 + \dots + G'_T,$$

where G'_1, \dots, G'_T are independent geometric random variables with parameter $1 - \frac{T}{d}$. Then,

$$\mathbb{P}(T_{v_0, v_1} + \dots + T_{v_{l-1}, v_l} \geq (1 + \epsilon)T) \leq \mathbb{P}(G'_1 + \dots + G'_T \geq (1 + \epsilon)T). \quad (3)$$

Now, using Lemma 2.3 with $C > 1 + \ln 2$ and n sufficiently large for $\frac{T}{d} < \delta$,

$$\mathbb{P}(G'_1 + \dots + G'_T \geq (1 + \epsilon)T) \leq \exp(-(C - 1)T),$$

as $T > \log_2 n$ we have that

$$\begin{aligned} \mathbb{P}(G'_1 + \dots + G'_T \geq (1 + \epsilon)T) &\leq \exp\left(-(C - 1)\frac{\ln n}{\ln 2}\right) \\ &= o(n^{-1}). \end{aligned}$$

Thus, $n \cdot \mathbb{P}(\mathcal{T}_{v, w'}(G) > (1 + \epsilon)T, \mathcal{J}(G) \leq T) = o(1)$ and the proof is finished. \square

The next lemma enables us to build a coupling between the PWR in G and the PWR in G_p .

Lemma 3.1 *Let G_n be a sequence of d_n -regular graphs, T_n and p_n as in Theorem 1.1. Take $C = \left(\frac{pd}{T}\right)^{\frac{1}{2}}$, then, for each $u \in V$, there exist random variables N_u^p with distribution $\text{Bin}(d, \frac{CT}{d})$, N_u with distribution $\text{Bin}(d, \frac{CT}{d}(1 - \frac{CT}{pd}))$, and random elements \mathcal{N}_u in $\Gamma(u)$, \mathcal{N}_u^p in $\Gamma_p(u)$ such that:*

$$\mathbb{P}(\mathcal{N}_u = S | N_u = k) = \frac{1}{\binom{d}{k}}, \text{ for each } S \subset \Gamma(u) \text{ with } |S| = k, \quad (4)$$

and

$$\mathbb{P}(\mathcal{N}_u^p = S' | N_u^p = j) = \frac{1}{\binom{\deg_p(u)}{j}}, \text{ for each } S' \subset \Gamma_p(u) \text{ with } |S'| = j. \quad (5)$$

Moreover, $\mathcal{N}_u \subset \mathcal{N}_u^p$ and there exists $\delta = \delta_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\mathbb{P}(|\mathcal{N}_u^p \setminus \mathcal{N}_u| \leq \delta \cdot |\mathcal{N}_u|, \forall u \in V) = 1 - o(n^{-1}). \quad (6)$$

Proof: Begin by taking independent random variables $\{I'_{u \rightarrow v} | (u, v) \in V^2, u \sim v\}$ with distribution $Be(\frac{CT}{pd})$, also take, for each $e \in e(G)$, independent random variables A_e with distribution $Be(p)$ (independent of the I 's).

Now defining, for each $(u, v) \in V^2$ with $u \sim v$, $I_{u \rightarrow v}^p = A_e I'_{u \rightarrow v}$ and, for each $u \in V$, $\mathcal{N}_u^p = \{v \sim u | I_{u \rightarrow v}^p = 1\}$ and $N_u^p = |\mathcal{N}_u^p|$, we have that $I_{u \rightarrow v}^p$ has distribution $Be(\frac{CT}{d})$ and therefore N_u^p has distribution $Bin(d, \frac{CT}{d})$ and \mathcal{N}_u^p satisfies (5).

We will use the following general fact to build N_u and \mathcal{N}_u :

Strassen's Lemma *Let μ and ν be distributions on \mathbb{R}^2 such that $\mu \preceq \nu$. Then, there exists a coupling (X, Y) of the (μ, ν) such that $X \leq Y$ in the coordinatewise partial order.*

Let μ be the law of $(I_{u \rightarrow v}^p, I_{v \rightarrow u}^p)$ and ν the distribution of (B_1, B_2) where B_1, B_2 are independent random variables $Be(q)$ such that $q = \frac{CT}{d}(1 - \frac{CT}{2pd})$. To show that $\nu \preceq \mu$, by the symmetry of the distributions, it suffices to show that $S_2 \preceq S_1$, where $S_1 = I_{u \rightarrow v}^p + I_{v \rightarrow u}^p$ and $S_2 = B_1 + B_2$.

Our choice of q implies, by simple calculations, that $\mathbb{P}(S_2 \geq 2) \leq \mathbb{P}(S_1 \geq 2)$ and $\mathbb{P}(S_2 \geq 1) \leq \mathbb{P}(S_1 \geq 1)$ so, $S_2 \preceq S_1$. Now, using Strassen's Lemma, we obtain that there exists $(I_{u \rightarrow v}, I_{v \rightarrow u})$ with distribution $Be(q) \times Be(q)$ and such that $I_{u \rightarrow v} \leq I_{u \rightarrow v}^p$ and $I_{v \rightarrow u} \leq I_{v \rightarrow u}^p$. Moreover, as

$$\mathbb{P}(I_{u \rightarrow v}^p - I_{u \rightarrow v} = 1) = \mathbb{E}[I_{u \rightarrow v}^p - I_{u \rightarrow v}] = \frac{1}{2p} \left(\frac{CT}{d} \right)^2$$

we have that $(I_{u \rightarrow v}^p - I_{u \rightarrow v})$ has distribution $Be\left(\frac{1}{2p} \left(\frac{CT}{d}\right)^2\right)$.

Defining, for each $u \in V$, $\mathcal{N}_u = \{v \sim u | I_{u \rightarrow v} = 1\}$ and $N_u = |\mathcal{N}_u|$. It immediately follows that N_u has distribution $Bin\left(d, \frac{CT}{d} \left(1 - \frac{CT}{2pd}\right)\right)$, \mathcal{N}_u^p satisfies (4) and $\mathcal{N}_u \subset \mathcal{N}_u^p$. It remains to prove (6).

Let us prove (6). Start by choosing $\delta = \max\left\{C^{-\frac{1}{2}}, \left(\frac{CT}{pd}\right)^{\frac{1}{2}}\right\}$ and defining $N_u^* = |\mathcal{N}_u^p \setminus \mathcal{N}_u|$ that has distribution $Bin\left(d, \frac{1}{2p} \left(\frac{CT}{d}\right)^2\right)$. Let A be the event $\{|N_u - \mathbb{E}[N_u]| \leq \frac{1}{2} \mathbb{E}[N_u]\}$. As $T \geq \log_2 n$ and $C \rightarrow \infty$ we have that $\mathbb{E}[N_u] \gg \ln n$. So, we can use Chernoff bounds for the binomial distribution to obtain $\mathbb{P}(A^c) = o(n^{-2})$. Thus,

$$\begin{aligned} \mathbb{P}(N_u^* > \delta N_u) &= \mathbb{P}(N_u^* > \delta N_u, A) + \mathbb{P}(A^c) \\ &\leq \mathbb{P}\left(N_u^* > \delta \frac{3\mathbb{E}[N_u]}{2}\right) + o(n^{-2}) \\ &= o(n^{-2}) \end{aligned}$$

where in the last inequality we use Chernoff bounds again and our choice of δ which ensures $\delta \cdot \mathbb{E}[N_u] \gg \max\{\ln n, \mathbb{E}[N_u^*]\}$. This finishes the proof of Lemma. \square

The next result studies the relation between $\mathcal{J}(G)$ and $\mathcal{J}(G_p)$.

Proposition 3.2 *Let $G_n = (V_n, E_n)$ be a sequence of d_n -regular graphs, with $n \rightarrow \infty$, that satisfies $\mathcal{J}(G_n) \leq T_n$ with high probability. Then, given $\epsilon > 0$ and choosing $p_n = o(1)$ such that $T_n = o(p_n \cdot d_n)$, we have that $\mathcal{J}(G_{n,p_n}) \leq (1 + \epsilon)T_n$ with high probability.*

Proof: We begin building a coupling between the PWR in the graph G and the PWR in the graph G_p using the previous lemma as follows. Take random variables N_u, N_u^p and random elements $\mathcal{N}_u, \mathcal{N}_u^p$ as in Lemma 3.1 and fix an arbitrary order $\{1, \dots, n\}$ of the vertices of G . For each $u \in V$, order $\Gamma(u)$ in the following way:

- order \mathcal{N}_u uniformly (from 1 to N_u);
- order $\Gamma(u) \setminus \mathcal{N}_u$ uniformly (from $N_u + 1$ to $\deg(u)$).

Equation (4) ensures that this is a uniform ordering of the neighbors of u . Denote by $Ord_u = Ord_u(w_1^u, \dots, w_d^u)$, where $w_1^u < \dots < w_d^u$ are the neighbors of u , the random vector built by the manner above. Run the PWR(G) as follows. If t is the first time that the vertex u receives information then in step $t + 1$ the vertex u informs $w_{i_1}^u$ if the i_1 -th coordinate of $Ord_u(w_1^u, \dots, w_d^u)$ is equal to 1, in step $t + 2$ the vertex u informs $w_{i_2}^u$ if the i_2 -th coordinate of $Ord_u(w_1^u, \dots, w_d^u)$ is equal to 2 and so on.

Let us run the PWR(G_p) similarly. For each $u \in V$, order $\Gamma_p(u)$ in the following way:

- order \mathcal{N}_u^p uniformly, but conditioned to coincide with the order of \mathcal{N}_u (from 1 to N_u^p);
- order $\Gamma(u) \setminus \mathcal{N}_u^p$ uniformly (from $N_u^p + 1$ to $\deg_p(u)$).

Equation (5) ensures that this is a uniform ordering of the neighbors of u . Denote by $Ord_u^p = Ord_u^p(\tilde{w}_1^u, \dots, \tilde{w}_{\deg_p(u)}^u)$, where $\tilde{w}_1^u < \dots < \tilde{w}_{\deg_p(u)}^u$ are the neighbors of u in $\Gamma_p(u)$, the random vector built by the manner above. Run the PWR(G_p) using $\{Ord_u^p\}_{u \in V}$.

Now we will prove that if $\mathcal{J}(G) \leq T$ with high probability then $\mathcal{J}(G_p) \leq (1 + \epsilon)T$ with high probability. Take $\epsilon > 0$, we have that

$$\mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T) = \mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T, A) + \mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T, A^c), \quad (7)$$

where A is the event $\{\mathcal{J}(G) \leq T\} \cap \{N_u > \frac{CT}{2}, |\mathcal{N}_u^p \setminus \mathcal{N}_u| \leq \delta \cdot |\mathcal{N}_u|, N_u^p > \frac{CT}{2}, \forall u \in V\}$ with $C = \left(\frac{pd}{T}\right)^{\frac{1}{2}}$ and $\delta = \delta_n$ as in Lemma 3.1. First, we analyze the term $\mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T, A^c)$:

$$\begin{aligned} \mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T, A^c) &\leq \mathbb{P}(\mathcal{J}(G) \geq T) \\ &\quad + \mathbb{P}\left(\exists u; N_u \leq \frac{CT}{2}\right) \\ &\quad + \mathbb{P}\left(\exists u; N_u^p \leq \frac{CT}{2}\right) \\ &\quad + \mathbb{P}(\exists u; |\mathcal{N}_u^p \setminus \mathcal{N}_u| > \delta \cdot |\mathcal{N}_u|). \end{aligned}$$

By hypothesis $\mathbb{P}(\mathcal{J}(G) \geq T) = o(1)$. Using Chernoff bounds for the binomial distribution, $T \geq \log_2 n$ and $C \rightarrow \infty$ we have that $\mathbb{P}(N_u \leq \frac{CT}{2}) = o(n^{-1})$ and that $\mathbb{P}(N_u^p \leq \frac{CT}{2}) = o(n^{-1})$. So,

$$\mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T, A^c) = o(1).$$

Next, let us estimate the other term in equation (7). For this, let $\mathcal{J}_{v,u}(G) = \min\{t \in \mathbb{N} | u \in J(t)\}$ be the first time step in which u is informed by $\text{PWR}(G)$ and let w' be the vertex such that

$$\mathbb{P}(\mathcal{J}_{v,w'}(G_p) > (1 + \epsilon)T, A) = \max_{u \in V} \mathbb{P}(\mathcal{J}_{v,u}(G_p) > (1 + \epsilon)T, A).$$

Then,

$$\begin{aligned} \mathbb{P}(\mathcal{J}(G_p) > (1 + \epsilon)T, A) &\leq \mathbb{P}(\exists u \in V; \mathcal{J}_{v,u}(G_p) > (1 + \epsilon)T, A) \\ &\leq n \cdot \mathbb{P}(\mathcal{J}_{v,w'}(G_p) > (1 + \epsilon)T, A), \end{aligned}$$

and the last expression is equal to

$$n \cdot \mathbb{E} [\mathbf{1}_A \mathbb{P}(\mathcal{J}_{v,w'}(G_p) > (1 + \epsilon)T | \{Ord_u\}_{u \in V}, \{N_u\}_{u \in V}, \{N_u^p\}_{u \in V})], \quad (8)$$

because A is $\sigma(\{Ord_u\}_{u \in V}, \{N_u\}_{u \in V}, \{N_u^p\}_{u \in V})$ -measurable.

Defining

$$\mathbb{Q}(\cdot) = \mathbb{P}(\cdot | \{Ord_u\}_{u \in V} = \{(m_1^u, \dots, m_d^u)\}_{u \in V}, \{N_u\}_{u \in V} = \{n_u\}_{u \in V}, \{N_u^p\}_{u \in V} = \{n_u^p\}_{u \in V})$$

and μ as the law of $(\{Ord_u\}_{u \in V}, \{N_u\}_{u \in V}, \{N_u^p\}_{u \in V})$, we obtain that expression (8) is equal to

$$n \cdot \int \mathbf{1}_{\tilde{A}}(\{\tilde{m}_u\}_u, \{n_u\}_u, \{n_u^p\}_u) \cdot \mathbb{Q}(\mathcal{J}_{v,w'}(G_p) > (1 + \epsilon)T) d\mu(\{\tilde{m}_u\}_u, \{n_u\}_u, \{n_u^p\}_u). \quad (9)$$

where $\tilde{m}_u = (m_1^u, \dots, m_d^u)$ and \tilde{A} is a Borel measurable set such that $\mathbf{1}_A$ is equal to $\mathbf{1}_{\tilde{A}}(\{Ord_u\}_u, \{N_u\}_u, \{N_u^p\}_u)$.

For $u \sim w$, let $J_{u,w}(G)$ be the time until vertex u chooses w to transmit the information according to the $\text{PWR}(G)$ and $J_{u,w}(G_p)$ the analogue for $\text{PWR}(G_p)$. Fixed $\{Ord_u = \tilde{m}_u\}_{u \in V}$, the event $\{\mathcal{J}(G) \leq T\}$ implies the existence of a path $\gamma : v = v_0 \sim \dots \sim v_l = w'$ such that $J_{v_0,v_1}(G) + \dots + J_{v_{l-1},v_l}(G) \leq T$. Also note that $N_{v_j} > T$ and $J_{v_j,v_{j+1}} \leq T$ implies $(v_j, v_{j+1}) \in G_p$ for each $j = 0, \dots, l-1$. So, expression (10) is less than or equal to

$$n \cdot \int \mathbf{1}_{\tilde{A}}(\{\tilde{m}_u\}_u, \{n_u\}_u, \{n_u^p\}_u) \cdot \mathbb{Q}(J_{v_0,v_1}(G_p) + \dots + J_{v_{l-1},v_l}(G_p) > (1 + \epsilon)T) d\mu. \quad (10)$$

Now, we will find the conditional distribution of $J_{v_j, v_{j+1}}(G_p)$. Take $0 \leq r \leq m \leq l$ and $0 \leq k \leq l$, we want to calculate

$$\mathbb{P}\left(J_{v_j, v_{j+1}}(G_p) = k | J_{v_j, v_{j+1}}(G) = r, N_{v_j} = m, N_{v_j}^p = l\right).$$

As $r \leq m$, we have that $v_{j+1} \in \mathcal{N}_{v_j} \subset \mathcal{N}_{v_j}^p$. Moreover, $J_{v_j, v_{j+1}}(G) = r$ implies that v_{j+1} is the $(j+1)$ -th vertex chosen in uniform ordering of \mathcal{N}_{v_j} . As the order of $\mathcal{N}_{v_j}^p$ preserves the order of \mathcal{N}_{v_j} , we have that $J_{v_j, v_{j+1}}(G_p)$ is the time until the $(j+1)$ -th success is obtained in repeated random sampling without replacement from a dichotomous population of size $N_{v_j}^p$ with N_{v_j} successes. Then, in $\{J_{v_j, v_{j+1}}(G) \leq N_{v_j}\}$,

$$\mathbb{P}\left(J_{v_j, v_{j+1}}(G_p) = k | J_{v_j, v_{j+1}}(G), N_{v_j}, N_{v_j}^p\right) = \mathbb{P}\left(NH(N_{v_j}^p, N_{v_j}, J_{v_j, v_{j+1}}(G)) = k\right).$$

So, the expression (10) can be written as

$$n \cdot \int \mathbf{1}_{\tilde{A}}(\{\tilde{m}_u\}_u, \{n_u\}_u, \{n_u^p\}_u) \cdot \mathbb{P}(Y_0 + \dots + Y_l > (1 + \epsilon)T) d\mu \quad (11)$$

where, for each $j = 0, \dots, l-1$, Y_j has distribution $NH(N_{v_j}^p, N_{v_j}, J_{v_j, v_{j+1}}(G))$ and $J_{v_0, v_1} + \dots + J_{v_{l-1}, v_l} \leq T$ in \tilde{A} .

Inside $\{\forall u \in V; |\mathcal{N}_u^p \setminus \mathcal{N}_u| \leq \delta \cdot |\mathcal{N}_u|, N_u^p > \frac{CT}{2}\}$ we have that

$$\begin{aligned} \frac{N_u}{N_u^p} - \frac{J_{v_j, v_{j+1}}(G)}{N_u^p} &\geq \frac{N_u}{N_u + |\mathcal{N}_u^p \setminus \mathcal{N}_u|} - \frac{T}{N_u^p} \\ &\geq \frac{1}{1 + \delta} - \frac{2T}{CT} \\ &= 1 - o(1). \end{aligned}$$

Then, using previous section

$$\mathbb{P}(Y_0 + \dots + Y_l > (1 + \epsilon)T) \leq \mathbb{P}(G_0 + \dots + G_T > (1 + \epsilon)T)$$

where G_0, \dots, G_T are independent geometric random variables with parameter $1 - o(1)$. Thus, for n sufficiently large, we can use Lemma 2.3 and obtain that

$$\mathbb{P}(G_0 + \dots + G_T > (1 + \epsilon)T) = o(n^{-1}),$$

because $T \geq \log_2 n$. So, expression (11) is bounded by $o(1)$ and this finishes the proof of the proposition. \square

Proof: [Theorem 1.1] By hypothesis, $\mathcal{T}(G_n) \leq T_n$ with high probability. Then, using Proposition 3.1, we have that $\mathcal{J}(G_n) \leq T_n$ w.h.p.. As $T_n = o(p_n \cdot d_n)$ and $p_n = o(1)$ we can use Proposition 3.2 and obtain that $\mathcal{J}(G_{n, p_n}) \leq (1 + \epsilon')T_n$ w.h.p.. Finally, we use the second part of Proposition 3.1 to conclude that $\mathcal{T}(G_{n, p_n}) \leq (1 + \epsilon')^2 T_n$ w.h.p.. Choosing ϵ' appropriately we have the result. \square

4 Appendix

In this appendix we prove Lemma 2.2. First, consider the following theorem (see [6]):

Theorem 4.1 (Disintegration) *Let ξ and η be two random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a measurable function f on $\mathbb{R} \times \mathbb{R}$ with $\mathbb{E}[|f(\xi, \eta)|] < \infty$. Then*

$$\mathbb{E}[f(\xi, \eta)|\eta] = \int f(s, \eta) \mu(\eta, ds), \mathbb{P} - a.s.,$$

where $\mu(\eta, \cdot)$ is the regular conditional probability of ξ given η .

Proof: [Lemma 2.2] We proceed by induction in r .

For $r = 1$ the result follows because $X_1 - X_0|X_0 \sim X_1$ and $X_1 \preceq Y_1$.

Now, define $f(\xi, \eta) = 1\{\xi + \eta \geq t\}$. Hence

$$\begin{aligned} \mathbb{P}(X_r \geq t) &= \mathbb{P}(X_{r-1} + X_r - X_{r-1} \geq t) \\ &= \mathbb{E}[f(X_{r-1}, X_r - X_{r-1})] \\ &= \mathbb{E}[\mathbb{E}[f(X_{r-1}, X_r - X_{r-1})|X_{r-1}]] \end{aligned}$$

Let $\mu(X_{r-1}, \cdot)$ be the regular conditional probability of $X_r - X_{r-1}$ given X_{r-1} and ν the distribution of Y_r . By Theorem 4.1, the expression above is equal to

$$\mathbb{E} \left[\int f(X_{r-1}, s) \mu(X_{r-1}, ds) \right].$$

By assumption, we have $X_r - X_{r-1}|X_{r-1} \preceq Y_r$ and this means that $\mu(X_{r-1}, \cdot) \preceq \nu$, \mathbb{P} -almost surely. As $f(\xi, \cdot)$ is increasing upper semicontinuous we have that

$$\mathbb{E} \left[\int f(X_{r-1}, s) \mu(X_{r-1}, ds) \right] \leq \mathbb{E} \left[\int f(X_{r-1}, s) \nu(ds) \right].$$

Now, take $\tilde{Y}_r \sim Y_r$ independent of X_{r-1} and let ϑ be the distribution of \tilde{Y}_r . The last expression is equal to

$$\mathbb{E} \left[\int f(X_{r-1}, s) \vartheta(ds) \right] = \mathbb{P}(X_{r-1} + \tilde{Y}_r \geq t).$$

Since \tilde{Y}_r is independent of X_{r-1} , we can use the substitution principle to obtain

$$\begin{aligned} \mathbb{P}(X_{r-1} + \tilde{Y}_r \geq t) &= \mathbb{E} \left[\mathbb{E}[f(X_{r-1}, \tilde{Y}_r)|\tilde{Y}_r] \right] \\ &= \mathbb{E} \left[\mathbb{E}[f(X_{r-1}, \tilde{y}_r)](\tilde{y}_r = \tilde{Y}_r) \right] \\ &= \mathbb{E} \left[\mathbb{P}(X_{r-1} \geq t - \tilde{y}_r)(\tilde{y}_r = \tilde{Y}_r) \right]. \end{aligned}$$

By the induction hypothesis, there exist $\tilde{Y}_i \sim Y_i$, for $i = 1, \dots, r-1$, independent random variables such that $X_{r-1} \preceq \tilde{Y}_1 + \dots + \tilde{Y}_{r-1}$ (we can take $\tilde{Y}_1, \dots, \tilde{Y}_{r-1}$ independent of \tilde{Y}_r). Therefore,

$$\begin{aligned} \mathbb{P}(X_r \geq t) &\leq \mathbb{P}(X_{r-1} + \tilde{Y}_r \geq t) \\ &\leq \mathbb{P}(\tilde{Y}_1 + \dots + \tilde{Y}_{r-1} + \tilde{Y}_r \geq t) \end{aligned}$$

and the result follows. \square

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References

- [1] N. Alon and J. Spencer, *The Probabilistic Method*, second ed., Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley and Sons, 2008.
- [2] F. Chierichetti, S. Lattanzi, and A. Panconesi, *Almost tight bounds on rumour spreading by conductance*, 41st STOC, 2010.
- [3] U. Feige, D. Peleg, P. Raghavan, and E. Upfal, *Randomized broadcast in networks*, Random Structures and Algorithms, 1990.
- [4] N. Fountoulakis, A. Huber, and K. Panagiotou, *Reliable Broadcasting in Random Networks and the Effect of Density*, INFOCOM, 2010.
- [5] A. Frieze and G. Grimmett, *The shortest-path problem for graphs with random arc-lengths*, Discrete Appl. Math., 1985.
- [6] O. Kallenberg, *Foundations of Modern Probability*, second ed., Probability and its applications, Springer-Verlag, 2010.
- [7] T. Sauerwald and A. Stauffer, *Rumor spreading and vertex expansion on regular graphs*, SODA, 2011.

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